Grade 11/12 Math Circles

November 23, 2022

Generating Functions - Solutions

Exercise Solutions

Exercise 1

How many ways can we make 50 cents using just 5, 10 and 25 cent coins?

Exercise 1 Solution

There are 10 ways. If (a, b, c) represents choosing a 5 cent, b 10 cent and c 25 cent coins, the following are the possibilities: (10, 0, 0), (8, 1, 0), (6, 2, 0), (5, 0, 1), (4, 3, 0), (3, 1, 1), (2, 4, 0), (1, 2, 1), (0, 5, 0) and (0, 0, 2).

Exercise 2

How many ways can we roll the dice to get a sum of 9? What are those ways?

Exercise 2 Solution

There are 4 ways. Roll 3 then 6, 6 then 3, 4 then 5 and 5 then 4.

Exercise 3

What is the generating function for the combinatorial class with a set of length three 0-1 strings and weight function $w(x)$ = the number of 1's in the string?

Exercise 3 Solution

The 0-1 strings of length 3 are 000, 001, 010, 100, 110, 101, 011 and 111. One string has weight 0, three have weight 1, three have weight 2 and one has weight 3. So, the generating function is $1 + 3z + 3z^2 + z^3$.

Problem Set Solutions

1. Create combinatorial classes (including a set and weight function) to represent the following:
   (a) Rolling a regular 8-sided die.
   (b) Choosing some number of 5 cent coins from an infinitely large pile.
   (c) Drawing a card from a deck with four of each numbered card 1-10.

Solution: Note that combinatorial classes with a given set are not unique. The following include examples of valid weight functions for the situations.

(a) **Set:** ways to roll the die $\{1, 2, 3, 4, 5, 6, 7, 8\}$.
Weight Function: $w(x) = x$, the value shown on the die.

(b) **Set:** the ways that we could choose some number of coins $\{0, 1, 2, 3, 4, \ldots\}$.
Weight Function: $w(x) = 5x$, the total value of the coins chosen.

(c) **Set:** the ways a card can be drawn $\{1_a, 1_b, 1_c, 1_d, 2_a, 2_b, 2_c, 2_d, \ldots, 10_a, 10_b, 10_c, 10_d\}$.
The subscripts distinguish between the four cards with the same number.
Weight Function: $w(x) =$ value shown on the card. For example $w(1_c) = 1$.

2. Find the generating functions for the following combinatorial classes:
   (a) 0-1 strings of length two with a weight function of the number of 1's in the string.
   (b) One roll of 2 regular 4-sided dice, where the weight function is the sum of the values rolled.
   (c) 0-1 strings of length 4 with a weight function of the number of occurrences of 01.

Solution:

(a) The four 0-1 strings of length 2 are 00, 01, 10 and 11. One string has no 1's, two have one 1 and one has two 1's. Thus, the generating function is $1 + 2z + z^2$.

(b) Rolling two regular four-sided dice gives the potential results:

	1	**2**	**3**	**4**
1	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8

This gives a generating function of $z^2 + 2z^3 + 3z^4 + 4z^5 + 3z^6 + 2z^7 + z^8$.

(c) There are 16 0-1 strings of length 4. Five such strings, 0000, 1000, 1100, 1110 and 1111, have no instances of 01. Ten have one instance: 0001, 0010, 0100, 0011, 1001, 0110, 1010, 0111, 1011 and 1101. Finally, one has two occurrences, 0101. This gives a generating function of $5 + 10z + z^2$.

3. For $F(z) = 1 + 3z + 6z^2 + 10z^3 + 15z^4 + 21z^5 + 28z^6 + \ldots$, determine:

(a) $[z^0]F(z)$

(b) $[z^3]F(z)$

(c) $[z^9]F(z)$

(d) $[z^n]F(z)$

Solution:

(a) $[z^0]F(z) = 1$

(b) $[z^3]F(z) = 10$

(c) $[z^9]F(z) = 28 + 8 + 9 + 10 = 55$ (or $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$).

(d) $[z^n]F(z) = 1 + 2 + \ldots + (n+1) = \frac{(n+1)(n+2)}{2}$

4. Write the following generating functions as simply as possible:

(a) $F(z) = 2z^2 + 4z^3 + 8z^4 + 16z^5 + 32z^6 + \ldots$

(b) $G(z) = 1 + z^2 + z^4 + z^6 + z^8 + \ldots$

(c) $H(z) = z + 3z^3 + 9z^5 + 27z^7 + 81z^9 + \ldots$

Solution:

(a)

$$\begin{aligned} F(z) &= 2z^2 + 4z^3 + 8z^4 + 16z^5 + 32z^6 + \ldots \\ &= 2z^2(1 + 2z + 4z^2 + 8z^3 + 16z^4 + \ldots) \\ &= 2z^2 \sum_{n=0}^{\infty} (2z)^n \\ &= \frac{2z^2}{1 - 2z} \end{aligned}$$

(b)

$$\begin{aligned} G(z) &= 1 + z^2 + z^4 + z^6 + z^8 + \dots \\ &= \sum_{n=0}^{\infty} z^{2n} \\ &= \sum_{n=0}^{\infty} (z^2)^n \\ &= \frac{1}{1-z^2} \end{aligned}$$

(c)

$$\begin{aligned} H(z) &= z + 3z^3 + 9z^5 + 27z^7 + 81z^9 + \dots \\ &= z(1 + 3z^2 + 9z^4 + 27z^6 + 81z^8 + \dots) \\ &= z\sum_{n=0}^{\infty} 3^n z^{2n} \\ &= z\sum_{n=0}^{\infty} (3z^2)^n \\ &= \frac{z}{1-3z^2} \end{aligned}$$

5. Find a generating function for 0-1 strings beginning with 11 and ending with 0, where the weight function is the length of the string.

Solution:
Define $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ as the following:

$\mathcal{A}$:
Set: 0-1 strings
Weight Function: Length of string
GF: $A(z) = \frac{1}{1-2z}$

$\mathcal{B}$:
Set: $\{11\}$
Weight Function: Length of string
GF: $B(z) = z^2$

$\mathcal{C}$:
Set: $\{0\}$
Weight Function: Length of string
GF: $C(z) = z$

This gives:
$\mathcal{B} * \mathcal{A} * \mathcal{C}$:
Set: $(11, a, 0)$, where a is a 0-1 string
Weight Function: $w((11, a, 0)) = 2 + w_A(a) + 1$
Generating Function: $B(z)A(z)C(z) = \frac{z^3}{1-2z} = z^3 + 2z^4 + 4z^5 + 8z^6 + 16z^7 + \ldots$

6. Consider the combinatorial class of 0-1 strings beginning with 101 and ending with 10, with a weight function corresponding to the length of the string.

   (a) Using a method similar to *Example: Multiplication 2* from the lesson, find a generating function for this combinatorial class.

   (b) How many 0-1 strings of length four start with 101 and end in 10? Is the actual answer the same as in the generating function from part a?

   (c) **Challenge:** If they are different, why might that be? How could it be fixed?

Solution:

(a) Define $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ as the following:

$\mathcal{A}$:
Set: 0-1 strings
WF: Length of string
GF: $A(z) = \frac{1}{1-2z}$

$\mathcal{B}$:
Set: $\{101\}$
WF: Length of string
GF: $B(z) = z^3$

$\mathcal{C}$:
Set: $\{10\}$
WF: Length of string
GF: $C(z) = z^2$

This gives:
$\mathcal{B} * \mathcal{A} * \mathcal{C}$:
Set: $(101, a, 10)$, where a is a 0-1 string
Weight Function: $w((101, a, 10)) = 3 + w_A(a) + 2$
Generating Function: $B(z)A(z)C(z) = \frac{z^5}{1-2z} = z^5 + 2z^6 + 4z^7 + 8z^8 + 16z^9 + \ldots$

(b) There is one 0-1 string of length 4, 1010, which starts with a 101 and ends in 10. This is not the same as from the generating function, which gives $[z^4]F(z) = 0$.

(c) They are different as 101 and 10 can overlap. By constructing our generating function as in part a, we don't allow for any overlapping as, at least, there is the empty string between the start and end. To fix this, we can add z^4 to our generating function. Since, if they overlap, they must have length 4, $F(z) + z^4$ is the correct generating function.

7. Find an expression for the number of ways to make \$2.65 in change (with 5, 10 and 25 cent coins available, as well as 1 and 2 dollar coins).

Solution: We answer this in a similar manner to the coin problem. Define the following:

$\mathcal{A}$:
Set:
Choices for the number of 5¢ coins, $\{0, 1, 2, \ldots\}$
WF: $w_A(x) = 5x$
GF: $A(z) = \frac{1}{1-z^5}$

$\mathcal{B}$:
Set:
Choices for the number of 10¢ coins, $\{0, 1, 2, \ldots\}$
WF: $w_B(x) = 10x$
GF: $B(z) = \frac{1}{1-z^{10}}$

$\mathcal{C}$:
Set:
Choices for the number of 25¢ coins, $\{0, 1, 2, \ldots\}$
WF: $w_C(x) = 25x$
GF: $C(z) = \frac{1}{1-z^{25}}$

$\mathcal{D}$:
Set:
Choices for the number of \$1 coins, $\{0, 1, 2, \ldots\}$
WF: $w_D(x) = 100x$
GF: $D(z) = \frac{1}{1-z^{100}}$

$\mathcal{E}$:
Set:
Choices for the number of \$2 coins, $\{0, 1, 2, \ldots\}$
WF: $w_E(x) = 200x$
GF: $E(z) = \frac{1}{1-z^{200}}$

Then,
$\mathcal{A} * \mathcal{B} * \mathcal{C} * \mathcal{D} * \mathcal{E}$:
Set: (a, b, c, d, e). In other words, each element represents a choice of coins.
Weight Function: $w((a, b, c, d, e)) = 5a + 10b + 25c + 100d + 200e$. In other words, the total value of the coins chosen.
Generating Function: $A(z)B(z)C(z)D(z)E(z) = \frac{1}{(1-z^5)(1-z^{10})(1-z^{25})(1-z^{100})(1-z^{200})}$
To get our required result, we need to find the coefficient of z^{265}, as \$2.65 is 265 cents. So, an expression for the number of ways to make \$2.65 in change is $[z^{265}]\frac{1}{(1-z^5)(1-z^{10})(1-z^{25})(1-z^{100})(1-z^{200})}$. *Note: Using SageMath, we find there are 258 ways.*

8. Find 2 four-sided dice such that:

- Each side has a positive integer number of dots
- The two dice are not the same
- The probability of rolling a sum of $2, \ldots, 8$ on these dice is the same as the probabilities for regular four-sided dice

Hint: $(z + z^2 + z^3 + z^4)^2 = (z^2 + 1)^2(z + 1)^2 z^2$

Solution: As with the Sicherman Dice, we start by converting each of these requirements into generating function terminology.

(a) Having a positive integer number of dots means that the new dice cannot have a constant (z^0) term.

(b) To have different dice, the corresponding generating functions for each die must be different.

(c) To have the same probabilities, the generating function created by multiplying the two dice generating functions together must be the same as for regular dice.

(d) To be four-sided, $A(1) = B(1) = 4$.

From Problem 2b, we found that the generating function of rolling two 4-sided dice is $F(z) = z^2 + 2z^3 + 3z^4 + 4z^5 + 3z^6 + 2z^7 + z^8 = (z + z^2 + z^3 + z^4)^2$. If we factor this, we get $F(z) = (z^2 + 1)^2(z + 1)^2z^2$.

In order to satisfy Requirement (a), we need each die to have a factor of z. Since we need the dice to be 4-sided and $(1^2 + 1) = (1 + 1) = 2$, we need each die to have two total of the $z^2 + 1$ and $z + 1$ factors. Since the dice cannot be the same, this gives new dice with generating functions $A(z) = z(z+1)^2 = z+2z^2+z^3$ and $B(z) = z(z^2+1)^2 = z+2z^3+z^5$. So, our two new dice have sides $\{1, 2, 2, 3\}$ and $\{1, 3, 3, 5\}$.

9. **Challenge:** Find a generating function for the following combinatorial class:
Set: Ordered lists of length three, where each entry is a positive integer.
Weight Function: Sum of the integers in the given ordered list.
Example: (2,3,1) has a weight of 2 + 3 + 1 = 6.

Solution: Since the elements of our set are ordered lists, we can break this into the product of three identical combinatorial classes, $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ with a set of positive integers and weight function equal to the value of the integers.

Then, $\mathcal{A} * \mathcal{B} * \mathcal{C}$ has the set and weight function of the given combinatorial class. The generating function for our identical classes is $z + z^2 + z^3 + z^4 + \ldots = \frac{z}{1-z}$. Hence, our generating function is $\frac{z^3}{(1-z)^3}$.